MATH 3060 Tutorial 9

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November 24, 2021

1 Questions

1. True or False: (a,b,c is related to Baire catgeory theorem)

   (a) There exists a metric d on $\mathbb{Q}$ so that d is complete and d and d_{std} define the same open sets.

   (b) $\mathbb{Q}$ is an intersection of countably many open subsets of $\mathbb{R}$.

   (c) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that for all $x > 0$, $\lim_{n\to\infty} f(nx) = 0$ ($n \in \mathbb{N}$). Then $\lim_{x\to\infty} f(x) = 0$.

   (d) Let $f : X \to Y$ be a continuous map of metric spaces, and S is a compact subset of X. Then $f(S)$ is compact.

2. (nested interval) Let X be a metric space. Show that a subset $S \subset X$ satisfies the following condition if and only if it is compact:
   For any decreasing sequence of closed subsets in S:

   $$S \supset E_1 \supset E_2 \supset E_3 \supset \cdots$$

   If each E_i is nonempty, then $\cap_{i=1}^{\infty} E_i$ is nonempty.

3. (An application of Inverse function theorem) Consider the function $p : \mathbb{R}^n \to \mathbb{R}^n$ given by

   $$p(x) = (p_1(x), p_2(x), \ldots, p_n(x)),$$

   where $p_k(x) = x_1^k + x_2^k + \cdots x_n^k$.

   (a) Show that p is not a local diffeomorphism near any point on the plane $x_1 = x_2$

   (b) Show that the Jacobian

   $$J = \det\left(\frac{\partial p_i}{\partial x_j}\right) = \prod_{i<j}(x_i - x_j)$$

   (c) What if we replace p_i by the elementary symmetric polynomials?

Proof. (a) Since $p(x+\epsilon, x, x_3, \dots) = p(x, x+\epsilon, x_3, \dots)$, p is not injective in any neighbourhood of $(x, x, x_3, \dots)$, and in particular not local diffeomorphism near that point.

(b) (a) part tells us that J vanishies on the plane $x_1 - x_2 = 0$. On the other hand, we know that J is a polynomial of degree $\frac{n(n-1)}{2}$, so $(x_1 - x_2)$ must be a factor of J. Similarly, $(x_i - x_j)$ is a factor of of J. Therefore, we must have

$$J = c \prod_{i<j} (x_i - x_j)$$

for some constant c. By comparing the coefficents of $x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$, we see that $c = 1$.

(c) Similar to (b), but $c = (-1)^{n(n-1)/2}$ this time.

□

4. (Baire Category theorem)

   (a) Show that there exists a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at every irrational number but discontinuous at every rational number.

   (b) Show that there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at every rational number but discontinuous at every irrational number.